

Inclusive semi-leptonic B decays to order $1/m_b^4$

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ABSTRACT: We give a systematic way to compute higher orders in the $1/m_b$ expansion in inclusive semi-leptonic decays at tree-level. We reproduce the known $1/m_b^3$ terms and compute the $1/m_b^4$ terms at tree-level. The appearing non-perturbative parameters and the impact on the determination of V_{cb} are discussed.

KEYWORDS: Heavy Quark Physics, B-Physics, Weak Decays.

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1. Introduction

Inclusive semi-leptonic decays are the cleanest way to access the matrix elements of the CKM matrix. In order to achieve precision in the determination of the CKM parameters a reliable theoretical framework is needed in addition to precise data. From the off-diagonal CKM matrix elements only V_{us} and V_{cb} are known very precisely from direct measurements [1], and both determinations can be related to a clean theoretical treatment: while the theoretical machinery for the determination of V_{us} from $K \rightarrow \pi \ell \bar{\nu}_\ell$ decays is chiral perturbation theory, the theoretical basis for the precision determination of V_{cb} is the Heavy Quark Expansion (HQE).

The HQE [2–5] has become a very reliable theoretical tool, in particular for semi-leptonic decays. At present, data on inclusive semi-leptonic decays $B \rightarrow X \ell \bar{\nu}_\ell$ are so precise that not only a determination of V_{cb} at the level of 2% accuracy is possible [6–9], but also a check of the consistency of the HQE can be performed by determining the parameters of the HQE in different ways [10]. These parameters are the kinetic energy parameter μ_π and the chromo-magnetic moment μ_G at order $1/m_b^2$ and the Darwin term ρ_D and the spin-orbit term ρ_{LS} at order $1/m_b^3$. We shall stick here with the kinetic scheme, the alternative $1S$ scheme [8, 9] yields similar precision. Using the moments of the hadronic invariant mass spectrum and the charged lepton energy spectrum these parameters can be consistently determined with an accuracy of about ten percent.

Within the HQE the order of the moments is related to the order in the $1/m_b$ expansion [11, 12]. For example, the moments $\langle (m_b - 2E_\ell)^n \rangle$ for the case of charmless semi-leptonic B decays are determined by the contributions of the order $1/m_b^n$. Thus in order to exploit precise measurements of the lepton energy spectrum, i.e. measurements of higher moments, it is mandatory to perform the theoretical calculation up to a sufficient order in the $1/m_b$ expansion.

The current theoretical state of the art which is used in the fits is already quite elaborate. At leading order (which is the partonic rate) the full $\mathcal{O}(\alpha_s)$ and the partial $\mathcal{O}(\alpha_s^2)$ result is known, while the $1/m_b^2$ and $1/m_b^3$ [13] results are used at tree-level. Hence the leading uncertainties are the $\mathcal{O}(\alpha_s)$ contributions at $1/m_b^2$ and the tree-level contributions to order $1/m_b^4$.

In the present paper we present a systematic way to perform the calculation of higher order terms in the $1/m_b$ expansion. This approach is then used to perform the complete calculation of the $1/m_b^4$ terms for the semi-leptonic decay $B \rightarrow X \ell \bar{\nu}_\ell$, keeping the charm mass to all orders. In the next section we shall outline our calculational method, which amounts to a systematization for the tree-level calculation of terms to some order $1/m_b^n$. This method is then applied to the case of $1/m_b^4$. In section 3 we identify and discuss the parameters appearing at $1/m_b^4$, where we find in total five independent matrix elements, which, however, have a simple physical interpretation. In section 4 we discuss the impact of these additional terms on the analysis of semi-leptonic decays.

2. Tree-level operator product expansion to $\mathcal{O}(1/m^n)$

In this section we outline a method which allows us - at least in principle - to calculate the decay rates of semi-leptonic inclusive decays at any order in the $1/m_b$ expansion at tree-level. The starting point for such a calculation is the hadronic tensor, which according to the optical theorem can be related to the discontinuity of a time-ordered product of currents across a cut. Thus one starts with a correlator of two hadronic currents

$$T_{\mu\nu} = \int d^4x e^{-ix(m_b v - q)} \langle B(p) | \bar{b}_v(x) \Gamma_\mu c(x) \bar{c}(0) \Gamma_\nu^\dagger b_v(0) | B(p) \rangle. \quad (2.1)$$

Here

$$\Gamma_\mu = \frac{1}{2} \gamma_\mu (1 - \gamma_5) \quad (2.2)$$

is the left-handed current, $v = \frac{p}{M_B}$ the four velocity of the decaying B meson and q the momentum transfer to the leptons.

It is convenient to decompose this correlator into Lorentz scalar structure functions according to

$$T_{\mu\nu} = -g_{\mu\nu}T_1 + v_\mu v_\nu T_2 - i\epsilon_{\mu\nu\alpha\beta}v^\alpha q^\beta T_3 + q_\mu q_\nu T_4 + (q_\mu v_\nu + v_\mu q_\nu) T_5 \quad (2.3)$$

where the scalar T_i are only functions of q^2 and vq .

Using the optical theorem we obtain for the relevant imaginary parts a similar decomposition:

$$W_{\mu\nu} = -g_{\mu\nu}W_1 + v_\mu v_\nu W_2 - i\epsilon_{\mu\nu\alpha\beta}v^\alpha q^\beta W_3 + q_\mu q_\nu W_4 + (q_\mu v_\nu + v_\mu q_\nu) W_5, \quad (2.4)$$

and the hadronic tensor, needed for the transition amplitude, can be computed from $T_{\mu\nu}$ by

$$-\frac{1}{\pi}\text{Im}T_j = W_j \quad (\text{left hand cut only}) \quad (2.5)$$

The differential decay rate is then obtained by contracting the hadronic tensor with the leptonic tensor $L^{\mu\nu}$, and one obtains

$$d\Gamma = \frac{G_F^2 |V_{cb}|^2}{4M_B} \text{Im}T_{\mu\nu} L^{\mu\nu} d\phi_{\text{PS}} \quad (2.6)$$

where $d\phi_{\text{PS}}$ denotes the corresponding phase space element.

Using the charged lepton energy $y = 2\frac{E_l}{m_b}$, the leptonic invariant mass $\hat{q}^2 = q^2/m_b^2$ and the rescaled total lepton energy $s = m_b v \hat{q}$ as independent variables one obtains the triple differential decay rate in terms of the scalar functions W_i

$$\begin{aligned} \frac{d^3\Gamma}{d\hat{q}^2 ds dy} &= \frac{G_F^2 m_b^2 |V_{cb}|^2}{2\pi^3} \theta\left(m_b^2\left(y\frac{2s}{m_b} - y^2 - \hat{q}^2\right)\right) \theta(\hat{q}^2) \\ &\times \left(W_1(\hat{q}^2, s) \hat{q}^2 + W_2(\hat{q}^2, s) \left(y\frac{s}{m_b} - \frac{y^2}{2} - \frac{\hat{q}^2}{2}\right) + W_3(\hat{q}^2, s) \hat{q}^2 (ym_b - s) \right) \end{aligned} \quad (2.7)$$

The tree-level expansion in $1/m_b$ is most easily set up by studying the Feynman diagram shown in figure 1. The double line denotes the propagator of a charm quark which is propagating in the background field of the soft gluons of the B meson. After rescaling the b quark momentum according to

$$p_b = m_b v + iD \quad (2.8)$$

we write for the background field propagator [14]

$$iS_{\text{BGF}} = \frac{1}{Q + i\not{D} - m_c} \quad (2.9)$$

where $Q = m_b v - q$ and D denotes the covariant derivative with respect to the background gluon field.

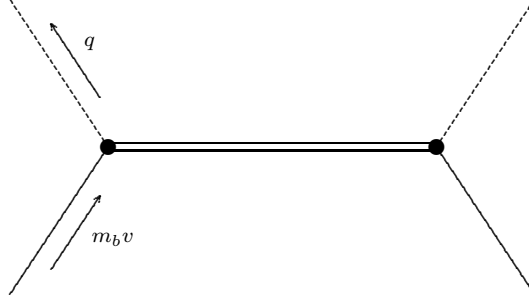


Figure 1: Tree-level Feynman diagram for the hadronic tensor in inclusive semi-leptonic B decays.

The tree-level OPE for semi-leptonic processes is obtained by multiplying (2.9) by the appropriate Dirac matrices for the left handed current (2.2). A calculation to order $1/m_b^n$ requires to expand this expression to n^{th} order in the covariant derivative (iD) according to

$$iS_{\text{BGF}} = \frac{1}{Q - m_c} - \frac{1}{Q - m_c} (i\not{D}) \frac{1}{Q - m_c} + \frac{1}{Q - m_c} (i\not{D}) \frac{1}{Q - m_c} (i\not{D}) \frac{1}{Q - m_c} + \dots \quad (2.10)$$

We note that this keeps track of the ordering of the covariant derivatives.

The remaining task is to evaluate the forward matrix elements of operators of the form

$$\bar{b}_{v,\alpha} (iD_{\mu_1}) \dots (iD_{\mu_n}) b_{v,\beta} \quad (2.11)$$

where \bar{b}_v (b_v) carries the spinor indices α (β). The field b_v is still the full QCD field, but redefined by a phase factor to remove the large piece of the b -quark momentum according to (2.8). This field satisfies the useful relations

$$b(x) = e^{-im_b v \cdot x} b_v(x) \quad (2.12)$$

$$\not{v} b_v = b_v - \frac{1}{m_b} i\not{D} b_v \quad (2.13)$$

$$P_+ b_v = -\frac{1}{2m_b} i\not{D} b_v + b_v \quad (2.14)$$

$$P_- b_v = \frac{1}{2m_b} i\not{D} b_v \quad (2.15)$$

$$(ivD) b_v = -\frac{1}{2m_b} i\not{D} i\not{D} b_v \quad (2.16)$$

which follow from the equation of motion for the b quark field. Here $P_{\pm} = (1 \pm \not{v})/2$ are the projectors on the “large” and “small” components of a Dirac spinor.

Note that this way of expanding will yield only matrix elements of local operators; however, these matrix elements contain still a nontrivial mass dependence, which will be discussed in the following.

The evaluation of these matrix elements is performed most conveniently in a recursive fashion. The starting point is always the matrix element of highest dimension, i.e. the one with the maximal number of covariant derivatives. For these matrix elements one may

neglect all contributions of order $1/m_b$ relative to this matrix element; in other words, this matrix element may be treated in the static limit.

Thus the first step is to consider the static limit of the forward matrix element of the highest dimensional operator, which has the form [15]

$$\begin{aligned} \langle B(p) | \bar{b}_{v,\alpha}(iD_{\mu_1}) \dots (iD_{\mu_n}) b_{v,\beta} | B(p) \rangle &= \langle B_v | \bar{h}_{v,\alpha}(iD_{\mu_1}) \dots (iD_{\mu_n}) h_{v,\beta} | B_v \rangle + \mathcal{O}(1/m_b) \\ &= 1_{\beta\alpha} A_{\mu_1\mu_2\dots\mu_n} + s_\lambda B_{\mu_1\mu_2\dots\mu_n}^\lambda \end{aligned} \quad (2.17)$$

where $s_\lambda = P_+ \gamma_\lambda \gamma_5 P_+$ is the generalization of the Pauli matrices to the case $v \neq (1, 0, 0, 0)$ and $|B_v\rangle$ is the static limit of the B meson state $|B(p)\rangle$.

The tensor structures A and B have to be related to a minimal set of fundamental matrix elements which are defined by contracting the indices in the various possible ways. In the following, these matrix elements are called *basic parameters* for a certain order in $1/m_b$. For example, at order $1/m_b^3$ these basic parameters are the Darwin term $\hat{\rho}_D$ and the spin-orbit term $\hat{\rho}_{LS}$ defined by

$$2M_B \hat{\rho}_D^3 = \langle B(p) | \bar{b}_v(iD_\mu)(ivD)(iD^\mu)b_v | B(p) \rangle \quad (2.18)$$

$$2M_B \hat{\rho}_{LS}^3 = \langle B(p) | \bar{b}_v(iD_\mu)(ivD)(iD_\nu)(-i\sigma^{\mu\nu})b_v | B(p) \rangle \quad (2.19)$$

while at order $1/m_b^2$ the basic parameters are the kinetic energy parameter $\hat{\mu}_\pi$ and the chromomagnetic moment $\hat{\mu}_G$ given by

$$2M_B \hat{\mu}_\pi^2 = -\langle B(p) | \bar{b}_v(iD)^2 b_v | B(p) \rangle \quad (2.20)$$

$$2M_B \hat{\mu}_G^2 = \langle B(p) | \bar{b}_v(iD_\mu)(iD_\nu)(-i\sigma^{\mu\nu})b_v | B(p) \rangle \quad (2.21)$$

The hat above the quantities means that we have defined these parameters in a covariant way. The definitions which have been recently used refer to the spatial components of the derivatives only and differ from the ones used here by higher-order terms in the $1/m_b$ expansion. The usual definitions of these quantities are given by

$$2M_B \mu_\pi^2 = -\langle B(p) | \bar{b}_v(iD_\perp)^2 b_v | B(p) \rangle \quad (2.22)$$

$$2M_B \mu_G^2 = \langle B(p) | \bar{b}_v(iD_\perp^\mu)(iD_\perp^\nu)(-i\sigma_{\mu\nu})b_v | B(p) \rangle \quad (2.23)$$

$$2M_B \rho_D^3 = \langle B(p) | \bar{b}_v(iD_{\perp\mu})(ivD)(iD_\perp^\mu)b_v | B(p) \rangle \quad (2.24)$$

$$2M_B \rho_{LS}^3 = \langle B(p) | \bar{b}_v(iD_\perp^\mu)(ivD)(iD_\perp^\nu)(-i\sigma_{\mu\nu})b_v | B(p) \rangle \quad (2.25)$$

where the spatial components are

$$D_\perp^\mu = (g^{\mu\nu} - v^\mu v^\nu) D_\nu \quad (2.26)$$

At $1/m_b^4$ the correction terms in the relations between the covariant and the usual definition do matter and are given in the next section.

Once the tensors A and B for the matrix elements of the highest order in the $1/m_b$ expansion have been expressed in terms of these fundamental parameters, one proceeds in

a similar way with the matrix elements of dimension $n - 1$. However, now we have to take into account all possible Dirac structures, such that

$$\langle B(p) | \bar{b}_{v,\alpha} (iD_{\mu_1}) \dots (iD_{\mu_{n-1}}) b_{v,\beta} | B(p) \rangle = \sum_i \hat{\Gamma}_{\beta\alpha}^{(i)} A_{\mu_1\mu_2\dots\mu_{n-1}}^{(i)} \quad (2.27)$$

where $\hat{\Gamma}^{(i)}$ are the complete set of the sixteen Dirac matrices. By using the fact that the relations (2.12), (2.16) connect different orders of the $1/m_b$ expansion we may express the tensor coefficients $A^{(i)}$ in terms of the basic parameters of the order $1/m_b^{n-1}$ and the ones of the order $1/m_b^n$.

This prescription defines a way to recursively compute the relevant matrix elements of the $1/m_b$ expansion up to order $1/m_b^n$ at tree-level, starting from the operator of highest dimension. Thus the leading matrix element of dimension three will then be expressed by this recursive method as a series in $1/m_b^n$ involving all the basic parameters up to this order.

Finally, the hadronic tensor is obtained from the trace formula

$$\begin{aligned} T_{\rho\sigma} &= \langle B(p) | \bar{b}_v \Gamma_\rho S_{\text{BGF}} \Gamma_\sigma^\dagger b_v | B(p) \rangle \\ &= \sum_i \text{Tr} \left\{ \Gamma_\rho \frac{1}{\not{Q} - m_c} \Gamma_\sigma^\dagger \hat{\Gamma}^{(i)} \right\} A^{(i,0)} \\ &\quad + \sum_i \text{Tr} \left\{ \Gamma_\rho \frac{1}{\not{Q} - m_c} \gamma^{\mu_1} \frac{1}{\not{Q} - m_c} \Gamma_\sigma^\dagger \hat{\Gamma}^{(i)} \right\} A_{\mu_1}^{(i,1)} \\ &\quad + \sum_i \text{Tr} \left\{ \Gamma_\rho \frac{1}{\not{Q} - m_c} \gamma^{\mu_1} \frac{1}{\not{Q} - m_c} \gamma^{\mu_2} \frac{1}{\not{Q} - m_c} \Gamma_\sigma^\dagger \hat{\Gamma}^{(i)} \right\} A_{\mu_1\mu_2}^{(i,2)} + \dots \end{aligned} \quad (2.28)$$

where the tree-level expansion of the background-field propagator (2.10) automatically yields the correct ordering of the covariant derivatives. Note that this saves us from computing the one- and even more-gluon matrix elements which would be necessary to obtain the correct ordering of covariant derivatives in the standard computation.

In the following we explicitly perform this recursion to order $1/m_b^4$, and we first identify the basic parameters for the order $1/m_b^4$.

3. Basic parameters at $1/m_b^4$

At $1/m_b^4$ we have to deal with operators of dimension 7, containing four covariant derivatives. We find in total five basic parameters at order $1/m_b^4$, three of which are spin independent, while two are spin dependent. Written in a covariant form we define the basic parameters of order $1/m_b^4$ to be

$$2M_B s_1 = \langle B(p) | \bar{b}_v iD_\rho (ivD)^2 iD^\rho b_v | B(p) \rangle \quad (3.1)$$

$$2M_B s_2 = \langle B(p) | \bar{b}_v iD_\rho (iD)^2 iD^\rho b_v | B(p) \rangle \quad (3.2)$$

$$2M_B s_3 = \langle B(p) | \bar{b}_v ((iD)^2)^2 b_v | B(p) \rangle \quad (3.3)$$

$$2M_B s_4 = \langle B(p) | \bar{b}_v iD_\mu (iD)^2 iD_\nu (-i\sigma^{\mu\nu}) b_v | B(p) \rangle \quad (3.4)$$

$$2M_B s_5 = \langle B(p) | \bar{b}_v iD_\rho iD_\mu iD_\nu iD^\rho (-i\sigma^{\mu\nu}) b_v | B(p) \rangle \quad (3.5)$$

which are real, since they are forward matrix elements of hermitean operators.

In order to obtain some intuition concerning the physical meaning of these parameters, we may relate them to some more intuitive quantities, which are

$$\langle \mathbf{E}^2 \rangle : \quad \text{Expectation value of the chromoelectric field squared} \quad (3.6)$$

$$\langle \mathbf{B}^2 \rangle : \quad \text{Expectation value of the chromomagnetic field squared} \quad (3.7)$$

$$\langle (\mathbf{p}^2)^2 \rangle : \quad \text{Fourth power of the residual } b \text{ quark momentum} \quad (3.8)$$

$$\langle (\mathbf{p}^2)(\mathbf{S} \cdot \mathbf{B}) \rangle : \quad \text{Mixed chromomagnetic moment and res. momentum squared} \quad (3.9)$$

$$\langle (\mathbf{p} \cdot \mathbf{B})(\mathbf{S} \cdot \mathbf{p}) \rangle : \quad \text{Mixed chromomagnetic field and res. helicity} \quad (3.10)$$

The basic parameters defined in (3.1)– (3.5) can be related to the intuitive quantities by¹

$$2M_B s_1 = -g^2 \langle \mathbf{E}^2 \rangle \quad (3.11)$$

$$2M_B s_2 = g^2 (\langle \mathbf{E}^2 \rangle - \langle \mathbf{B}^2 \rangle) + \langle (\mathbf{p}^2)^2 \rangle \quad (3.12)$$

$$2M_B s_3 = \langle (\mathbf{p}^2)^2 \rangle \quad (3.13)$$

$$2M_B s_4 = -3g \langle (\mathbf{S} \cdot \mathbf{B})(\mathbf{p}^2) \rangle + 2g \langle (\mathbf{p} \cdot \mathbf{B})(\mathbf{S} \cdot \mathbf{p}) \rangle \quad (3.14)$$

$$2M_B s_5 = -g \langle (\mathbf{S} \cdot \mathbf{B})(\mathbf{p}^2) \rangle \quad (3.15)$$

which also gives some information on the sign of the non-perturbative parameters.

These parameters have to be determined independently from the lower-order parameters; however, for a numerical estimate it is useful to note that some of these parameters can be estimated in naive factorization, such as

$$\langle \mathbf{B}^2 \rangle \sim (\langle (\mathbf{S} \cdot \mathbf{B}) \rangle)^2 = \mu_G^4 \quad (3.16)$$

$$\langle (\mathbf{p}^2)^2 \rangle \sim (\langle \mathbf{p}^2 \rangle)^2 = \mu_\pi^4 \quad (3.17)$$

$$\langle (\mathbf{p}^2)(\mathbf{S} \cdot \mathbf{B}) \rangle \sim \langle (\mathbf{p}^2)^2 \rangle \langle (\mathbf{S} \cdot \mathbf{B}) \rangle = \mu_G^2 \mu_\pi^2 \quad (3.18)$$

while the two remaining matrix elements do not have a simple interpretation in naive factorization. Based on the interpretation of the parameters s_i in terms of physical quantities we define a “guestimate” for the basic parameters s_i by

$$s_1 \sim -\frac{\rho_D^6}{\mu_\pi^2}, \quad s_2 \sim \frac{\rho_D^6}{\mu_\pi^2} - \mu_G^4 + \mu_\pi^4, \quad s_3 \sim \mu_\pi^4, \quad s_4 \sim s_5 \sim -\mu_G^2 \mu_\pi^2 \quad (3.19)$$

where at least the sign of the contributions should be correctly reproduced.

In this way all the basic matrix elements up to order $1/m_b^4$ have been identified and the remaining task is to express any matrix element in terms of these basic quantities according to (2.27). In appendix A we list all the necessary general matrix elements up to dimension seven in terms of the basic parameters $\hat{\mu}_\pi$, $\hat{\mu}_G$, $\hat{\rho}_D$, $\hat{\rho}_{LS}$ and $s_1 \dots s_5$.

¹The relations between the s_i and the intuitive quantities are not entirely unique. For example, $\langle (\mathbf{p}^2)^2 \rangle$ could also be defined by the completely symmetrized combination of the covariant derivatives

Finally, we may now consider the relation between the covariant definition of the basic parameters and the usual ones shown in (2.22)– (2.25). This relation is given by

$$\hat{\mu}_\pi^2 = \mu_\pi^2 + \frac{1}{8m_b^2}[s_2 + s_3 + 4s_5] \quad (3.20)$$

$$\hat{\mu}_G^2 = \mu_G^2 - \frac{1}{m_b}[\rho_D^3 + \rho_{LS}^3] - \frac{1}{4m_b^2}[s_2 + s_3 + 4s_5] \quad (3.21)$$

$$\hat{\rho}_D^3 = \rho_D^3 \quad (3.22)$$

$$\hat{\rho}_{LS}^3 = \rho_{LS}^3 - \frac{s_1}{m_b} \quad (3.23)$$

4. Results and discussion

The remaining task is to evaluate the scalar components of the hadronic tensor at tree-level using (2.28) and the formulae from the appendix. We first compute the correlator (2.1) involving the time-ordered product. From the scalar components shown in (2.3) we need only T_1 , T_2 and T_3 due to current conservation of the leptonic current. The resulting expressions up to order $1/m_b^4$ are tedious and are given in appendix B

Taking the imaginary part to obtain the components of $W_{\mu\nu}$ according to (2.5) we use the relation

$$-\frac{1}{\pi}\text{Im}\Delta_0^{n+1} = \frac{(-1)^n}{n!}\delta^{(n)}(m_b^2 - m_c^2 + q^2 - 2m_b vq) \quad (4.1)$$

from which we can obtain the triple differential rate up to $1/m_b^4$ at tree-level using (2.7). The resulting expressions of the double differential rate, the single differential rate and the total rate are quite lengthy and given completely in appendix C.

The relevant quantities for the experimental analysis and the determination of V_{cb} are the moments of the lepton energy spectrum and the moments of the hadronic invariant mass spectrum. In order to get a quantitative idea of the effect of the $1/m_b^4$ terms we consider the $1/m_b^4$ contributions to the moments, normalized to the partonic rate at tree-level. Thus we define

$$\delta^{(4)}\langle M_X^n \rangle = \frac{1}{\Gamma_0} \int dM_X M_X^n \int_{E_{\text{cut}}} dE_\ell \frac{d^2\Gamma^{(4)}}{dM_x dE_\ell} \quad (4.2)$$

$$\delta^{(4)}\langle E_\ell^n \rangle = \frac{1}{\Gamma_0} \int dM_X \int_{E_{\text{cut}}} dE_\ell E_\ell^n \frac{d^2\Gamma^{(4)}}{dM_x dE_\ell} \quad (4.3)$$

$$\Gamma_0 = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} (1 - 8\rho - 12\log(\rho)\rho^2 + 8\rho^3 - \rho^4) \quad (4.4)$$

where $\Gamma^{(4)}$ is the contribution of order $1/m_b^4$.

| $i \setminus n$ | 1 | 2 | 3 | 4 |
|-----------------|----------|---------|---------|---------|
| 1 | - 11.570 | - 6.314 | - 3.418 | - 1.845 |
| 2 | 2.073 | 1.076 | 0.556 | 0.288 |
| 3 | - 5.969 | - 2.801 | - 1.320 | - 0.620 |
| 4 | - 0.102 | - 0.126 | - 0.105 | - 0.074 |
| 5 | - 3.377 | - 1.042 | - 0.174 | 0.089 |

Table 1: Values of the coefficients $g_i^{(n)}$ with a weak dependence on E_{cut} . The values quoted are for $E_{\text{cut}} = 0.8 \text{ GeV}$.

| $i \setminus n$ | 2 | 3 | 4 |
|-----------------|---------|---------|---------|
| 1 | 6.214 | - 6.633 | - 1.322 |
| 2 | - 1.343 | 1.026 | 0.203 |
| 3 | 2.472 | 1.358 | - 0.033 |
| 4 | - 0.059 | 0.315 | 0.133 |
| 5 | - 0.377 | - 0.129 | 0.019 |

Table 2: Values of the coefficients $f_i^{(n)}$ with a weak dependence on E_{cut} . The values quoted are for $E_{\text{cut}} = 0.8 \text{ GeV}$.

We first study the dependence of the $\mathcal{O}(1/m_b^4)$ contributions to the moments on the different non-perturbative parameters s_i . We write the moments as

$$\delta^{(4)} \langle M_X^n \rangle = \sum_{i=1}^5 m_b^n f_i^{(n)}(E_{\text{cut}}) \frac{s_i}{m_b^4} \tag{4.5}$$

$$\delta^{(4)} \langle E_\ell^n \rangle = \sum_{i=1}^5 m_b^n g_i^{(n)}(E_{\text{cut}}) \frac{s_i}{m_b^4} \tag{4.6}$$

The contributions of order $1/m_b^4$ are strongly concentrated in the endpoint $E_\ell \sim (m_b^2 - m_c^2)/(2m_b)$ for the lepton energy. Hence the dependence on the cut-off energy of the coefficients is small and can be neglected, except for the functions $f_i^{(1)}$, corresponding to the first hadronic mass moment.

In tables 1 and 2 we tabulate the values of the coefficients in (4.5) and (4.6) for those coefficients which are practically independent of E_{cut} .

As pointed out above, only the coefficients $f_i^{(1)}$ of the first hadronic moment depend on E_{cut} in a substantial way. Figures 2 and 3 show the dependence of the five functions on the energy cut.

Finally we shall also study the overall effect of the $\mathcal{O}(1/m_b^4)$ contributions. In order to do this we have to insert numerical values for the nonperturbative parameters. We use the relations (3.19) and the values obtained from the fit in [10] for the lower order basic parameters. In table 3 we list the values from [10] for reference; in table 4 we list the values of the basic parameters at $1/m_b^4$ obtained from our gestimate. The errors for our parameters s_1, \dots, s_5 are calculated by error propagation from the ‘‘gestimate’’ (3.19) and we add a 30% systematical uncertainty for the naive factorization assumption.

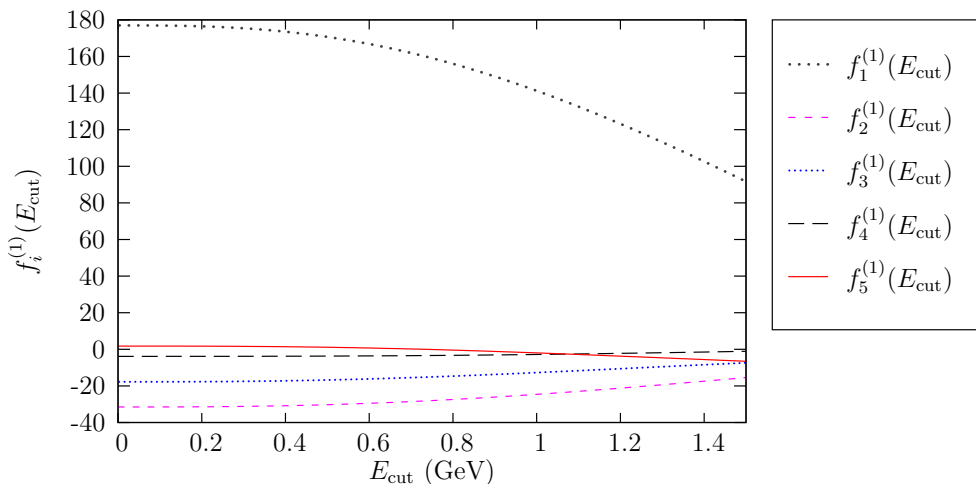


Figure 2: The dependence of $f_i^{(1)}$ on the energy cut in the lepton energy.

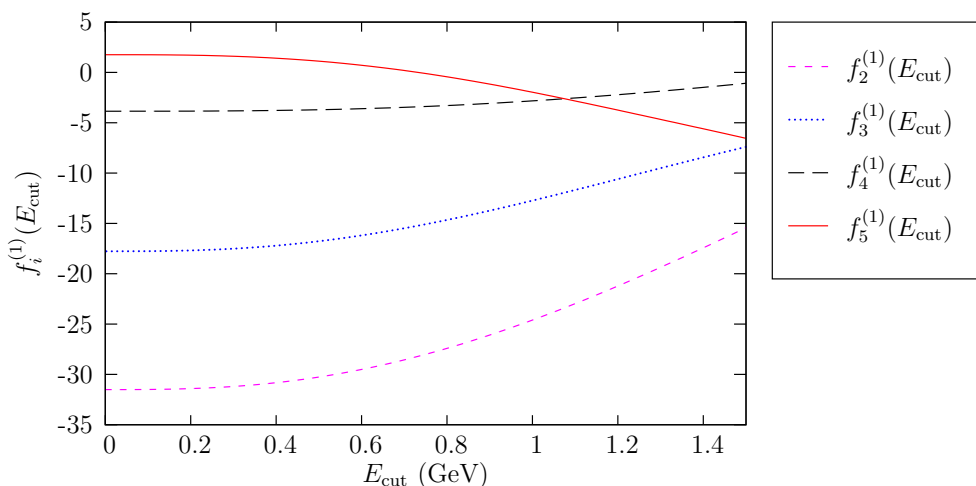


Figure 3: The dependence of $f_i^{(1)}$ on the energy cut in the lepton energy, leaving out the much larger function $f_1^{(1)}$.

| Parameter | μ_π^2 (GeV) ² | μ_G^2 (GeV) ² | ρ_{LS}^3 (GeV) ³ | ρ_D^3 (GeV) ³ |
|-----------|--------------------------------|------------------------------|----------------------------------|-------------------------------|
| Value | 0.40 ± 0.04 | 0.30 ± 0.05 | -0.18 ± 0.09 | 0.17 ± 0.02 |

Table 3: Values taken from Buchmüller and Flächer.

Inserting for the masses the values $m_b = 4.59$ GeV and $m_c = 1.142$ GeV from [10] we obtain for the contributions to the moments for a lepton energy cut of 0.8 GeV the values shown in table 5. As pointed out above, the $1/m_b^4$ contributions to the lepton energy moments are practically independent of E_{cut} , while the dependence on E_{cut} of the hadronic moments is given by the functions $f_i^{(1)}$.

| | | | | | |
|-----------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Parameter | s_1 (GeV) ⁴ | s_2 (GeV) ⁴ | s_3 (GeV) ⁴ | s_4 (GeV) ⁴ | s_5 (GeV) ⁴ |
| Value | -0.08 ± 0.03 | 0.15 ± 0.06 | 0.16 ± 0.06 | -0.12 ± 0.04 | -0.12 ± 0.04 |

Table 4: Our guess for the basic Parameters s_i which is used in the numerical analysis.

| | | | | |
|--------------------------------------|-------------|--------------------------|-------------------------|-------------------------|
| n | 1 | 2 | 3 | 4 |
| $\delta^{(4)} \langle M_X^n \rangle$ | -0.1835 GeV | -0.0104 GeV ² | 0.1850 GeV ³ | 0.1064 GeV ⁴ |
| $\delta^{(4)} \langle E_e^n \rangle$ | 0.0066 GeV | 0.0154 GeV ² | 0.0351 GeV ³ | 0.0803 GeV ⁴ |

Table 5: Numerical values for the contribution of order $1/m_b^4$ using the parameters from table 4.

The contribution to the total rate is very small; using our estimates for the parameters s_i we get

$$\frac{\delta^{(4)}\Gamma}{\Gamma} \approx 0.25\%$$

resulting in a completely negligible shift of the central value of V_{cb} from the terms of the order $1/m_b^4$.

5. Conclusions

In this paper we presented the complete result for the $\mathcal{O}(1/m_b^4)$ contributions to the semileptonic rate at tree-level. Although we have in total five new, non-perturbative parameters, we can use the known matrix elements to order $\mathcal{O}(1/m_b^3)$ to estimate the size of the contributions.

It turns out that the size of the $1/m_b^4$ terms is “normal”, i.e. we do not see any abnormally large coefficients in the $1/m_b$ expansion, at least in this tree-level calculation.

The formulae collected in the appendix allow us to include now the $\mathcal{O}(1/m_b^4)$ contributions into the moment analyses of semi-leptonic decays. In particular, the fourth central moment in M_X^2 is dominated by the $1/m_b^4$ terms, so these terms are required for a sensible analysis of higher-order moments.

The method suggested here allows in principle the calculation of arbitrarily high orders in the $1/m_b$ expansion at tree-level. However, the limitation is the number of basic parameters, which is presumably growing factorially, leading to a factorially growing effort to express the general matrix elements through the basic parameters.

On the other hand, assuming that the $1/m_b^{\geq 5}$ terms also behave “normally”, the uncertainty from these contributions in the determination of V_{cb} will really be extremely small. The uncertainty currently assigned to the use of the HQE for the determination of V_{cb} is about 1%; a conservative estimate is to use the size of the last term calculated as the uncertainty means that the uncertainty in the V_{cb} determination due to HQE may be reduced to $< 1\%$.

Acknowledgments

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A. General matrix elements in terms of basic parameters

In this appendix we list the general matrix elements up to dimension seven in terms of the parameters $\hat{\mu}_\pi$ (2.20), $\hat{\mu}_G$ (2.21), $\hat{\rho}_D$ (2.18), $\hat{\rho}_{LS}$ (2.19), and $s_1 \dots s_5$ (3.1)– (3.5). The results are written as Dirac matrices which have to be inserted into the trace formula (2.28). MATHEMATICA notebooks with the corresponding expressions can be obtained from the authors.

A.1 Dimension seven

$$\begin{aligned}
 \langle B(p) | \bar{b}_v (iD^\rho) (iD^\sigma) (iD^\lambda) (iD^\delta) b_v | B(p) \rangle = & \\
 \frac{M_B}{30} \mathbf{P}_+ \left\{ \frac{1}{2} \left((-i\sigma^{\lambda\rho}) (g^{\delta\sigma} - v^\delta v^\sigma) + (-i\sigma^{\delta\sigma}) (g^{\lambda\rho} - v^\lambda v^\rho) \right) (s_4 + s_5) \right. & \\
 + \left(-(-i\sigma^{\rho\sigma}) (g^{\delta\lambda} - v^\delta v^\lambda) + (-i\sigma^{\lambda\sigma}) (g^{\delta\rho} - v^\delta v^\rho) \right. & \\
 + (-i\sigma^{\delta\lambda}) (g^{\rho\sigma} - v^\rho v^\sigma) \left. \right) (4s_5 - s_4) & \\
 + (-i\sigma)^{\delta\rho} (g^{\lambda\sigma} - v^\lambda v^\sigma) (2s_4 - 3s_5) \left. \right\} \mathbf{P}_+ & \\
 + \frac{M_B}{60} \mathbf{P}_+ \left\{ (v^\delta v^\lambda - g^{\lambda\delta}) (g^{\rho\sigma} - v^\rho v^\sigma) (-2s_1 + 3s_2 - 7s_3) \right. & \\
 + 2 (g^{\delta\sigma} - v^\delta v^\sigma) (g^{\rho\lambda} - v^\lambda v^\rho) (s_1 + s_2 + s_3) & \\
 + (g^{\rho\delta} - v^\delta v^\rho) (g^{\sigma\lambda} (-8s_1 + 7s_2 - 3s_3) + v^\lambda v^\sigma (28s_1 - 7s_2 + 3s_3)) \left. \right\} & \\
 + \mathcal{O}(1/m_b) & \tag{A.1}
 \end{aligned}$$

A.2 Dimension six

$$\begin{aligned}
 \langle B(p) | \bar{b}_v (iD^\rho) (iD^\sigma) (iD^\lambda) b_v | B(p) \rangle = & \\
 \frac{M_B}{3} \mathbf{P}_+ \left\{ v^\sigma (g^{\rho\lambda} - v^\lambda v^\rho) \hat{\rho}_D^3 \right\} + \frac{M_B}{6} \mathbf{P}_+ \left\{ (-i\sigma^{\lambda\rho}) v^\sigma \hat{\rho}_{LS}^3 \right\} \mathbf{P}_+ & \\
 + \frac{M_B}{6m_b} \mathbf{P}_+ \left\{ (-i\sigma^{\lambda\rho}) v^\sigma s_1 \right\} \mathbf{P}_+ + \frac{M_B}{12m_b} \left\{ (-i\sigma^{\lambda\kappa}) v_\kappa \gamma^\rho v^\sigma s_1 \right\} & \\
 + \frac{M_B}{24m_b} \left\{ ((-i\sigma^{\rho\kappa}) v_\kappa v^\lambda v^\sigma - (-i\sigma^{\lambda\kappa}) v_\kappa v^\rho v^\sigma) (3s_1 - s_2 + s_3 - s_4 + s_5) \right\} & \\
 + \frac{M_B}{48m_b} \left\{ ((-i\sigma^{\rho\kappa}) v_\kappa v^\lambda - (i\sigma^{\lambda\kappa}) v_\kappa v^\rho) \gamma^\sigma (-2s_1 + s_2 - s_3 + 2s_5) \right\} &
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{M_B}{24m_b} P_+ \left\{ \left((-i\sigma^{\rho\sigma})v^\lambda - (-i\sigma^{\lambda\sigma})v^\rho \right) (-2s_1 + s_2 - s_3 + 2s_5) \right\} P_+ \\
 & + \frac{M_B}{24m_b} \left\{ \left((-i\sigma^{\rho\sigma})v^\lambda - (-i\sigma^{\lambda\sigma})v^\rho + (-i\sigma^{\lambda\rho})v^\sigma \right) \not\psi (-s_4 + 3s_5) \right\} \\
 & + \frac{M_B}{60m_b} \left\{ (\gamma^\sigma - v^\sigma \not\psi) (s_1 + s_2 + s_3 + s_4 + s_5) g^{\lambda\rho} \right\} \\
 & + \frac{M_B}{120m_b} \left\{ (\gamma^\rho \not\psi - v^\rho) \left((-6P_+ - 2)s_1 + (4P_+ + 3)s_2 - (7 - 4P_+)s_3 \right. \right. \\
 & \quad \left. \left. - 10s_5 + \frac{1}{2}(8P_+ + 6)(s_4 + s_5) \right) g^{\lambda\sigma} \right\} + \frac{M_B}{12m_b} \left\{ \gamma^\lambda \gamma^\rho \not\psi v^\sigma s_1 \right\} \\
 & + \frac{M_B}{120m_b} \left\{ \left((\gamma^\lambda \not\psi - v^\lambda) \left((8 - 6P_+)s_1 + (4P_+ + 3)s_3 + 10s_5 \right. \right. \right. \\
 & \quad \left. \left. - \frac{1}{2}(14 - 8P_+)(s_4 + s_5) \right) + (4P_+ + 3) (\not\psi \gamma^\lambda - v^\lambda) s_2 \right) g^{\rho\sigma} \right\} \\
 & + \frac{M_B}{6m_b} \left\{ v^\lambda v^\rho v^\sigma (s_3 + s_5) \right\} - \frac{M_B}{6m_b} P_+ \left\{ (v^\rho g^{\lambda\sigma} + v^\lambda g^{\rho\sigma}) (s_3 + s_5) \right\} \\
 & - \frac{M_B}{60m_b} \left\{ v^\lambda v^\rho \gamma^\sigma (s_1 + s_2 + s_3 + s_4 + s_5) \right\} \\
 & + \frac{M_B}{48m_b} \left\{ (\gamma^\rho \gamma^\sigma v^\lambda - \gamma^\lambda \gamma^\sigma v^\rho) \not\psi (-2s_1 + s_2 - s_3 + 2s_5) \right\} \\
 & + \frac{M_B}{120m_b} \left\{ v^\lambda v^\sigma \gamma^\rho (33s_1 - 7s_2 + 3s_3 - 10s_5 - 2(s_4 + s_5)) \right\} \\
 & + \frac{M_B}{120m_b} \left\{ v^\rho v^\sigma \gamma^\lambda (-17s_1 + 3s_2 - 7s_3 + 10s_5 - 2(s_4 + s_5)) \right\} \\
 & + \frac{M_B}{12m_b} \left\{ (\gamma^\rho \gamma^\sigma \gamma^\lambda + \gamma^\sigma g^{\rho\lambda} - \gamma^\lambda g^{\rho\sigma} - \gamma^\rho g^{\sigma\lambda}) \left(\frac{1}{2}(s_4 + s_5) - 2s_5 \right) \right\} \\
 & + \frac{M_B}{60m_b} \left\{ v^\lambda v^\rho v^\sigma \not\psi (-12s_1 + 3s_2 + 13s_3 + 3s_4 + 13s_5) \right\} \\
 & + \mathcal{O}(1/m_b^2)
 \end{aligned} \tag{A.2}$$

A.3 Dimension five

$$\begin{aligned}
 \langle B(p) | \bar{b}_v (iD^\rho) (iD^\sigma) b_v | B(p) \rangle = & \\
 & \frac{M_B}{24} P_+ \left\{ (g^{\rho\sigma} - v^\rho v^\sigma) \left(-8\hat{\mu}_\pi^2 - \frac{s_2 + s_3}{m_b^2} \right) \right\} \\
 & + \frac{M_B}{12} P_+ \left\{ (-i\sigma^{\rho\sigma}) \left(-2\hat{\mu}_G^2 + \frac{-2s_2 - s_4 + s_5}{2m_b^2} - \frac{2(\hat{\rho}_D^3 + \hat{\rho}_{LS}^3)}{m_b} \right) \right\} P_+ \\
 & + \frac{M_B}{48m_b^2} \left\{ \left((-i\sigma^{\rho\kappa})v_\kappa v^\sigma - (-i\sigma^{\sigma\kappa})v_\kappa v^\rho \right) (2s_3 - s_4 + 9s_5) \right\} \\
 & + \frac{M_B}{48m_b^2} \left\{ (-i\sigma^{\rho\sigma}) (s_2 - s_3 + s_4 - 5s_5) \right\} \\
 & + \frac{M_B}{6m_b} \left\{ (P_+ \gamma^\sigma v^\rho + \gamma^\rho P_+ v^\sigma) (\hat{\rho}_D^3 + \hat{\rho}_{LS}^3) \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{M_B}{24} P_+ \left\{ v^\rho v^\sigma \left(\frac{3s_2 + 3s_3 + 16s_5}{m_b^2} - \frac{8(\hat{\rho}_D^3 + \hat{\rho}_{LS}^3)}{m_b} \right) \right\} \\
 & + \frac{M_B}{48m_b^2} \left\{ (v^\sigma \gamma^\rho + v^\rho \gamma^\sigma) (4s_1 - s_2 - s_3 - 4s_5) \right\} + \frac{M_B}{12m_b^2} \left\{ \psi g^{\rho\sigma} (s_1 - s_2 - s_4) \right\} \\
 & + \frac{M_B}{6m_b^2} P_+ \left\{ g^{\rho\sigma} s_5 \right\} + \frac{M_B}{24m_b^2} \left\{ v^\rho v^\sigma \psi (-12s_1 + 3s_2 + s_3 + 2s_4 + 4s_5) \right\} \\
 & + \mathcal{O}(1/m_b^3)
 \end{aligned} \tag{A.3}$$

A.4 Dimension four

$$\begin{aligned}
 \langle B(p) | \bar{b}_v (iD^\rho) b_v | B(p) \rangle & = -\frac{M_B}{2m_b} P_+ \left\{ v^\rho (\hat{\mu}_G^2 - \hat{\mu}_\pi^2) \right\} + \frac{M_B}{6m_b} \left\{ (\gamma^\rho - v^\rho \psi) (\hat{\mu}_G^2 - \hat{\mu}_\pi^2) \right\} \\
 & + \frac{M_B}{12m_b^2} \left\{ (\gamma^\rho - 4 v^\rho \psi) (\hat{\rho}_{LS}^3 + \hat{\rho}_D^3) \right\} + \mathcal{O}(1/m_b^4)
 \end{aligned} \tag{A.4}$$

A.5 Dimension 3

$$\langle B(p) | \bar{b}_v b_v | B(p) \rangle = P_+ M_B + \frac{M_B}{4m_b^2} (\hat{\mu}_G^2 - \hat{\mu}_\pi^2) + \mathcal{O}(1/m_b^5) \tag{A.5}$$

B. Scalar components of the time-ordered product

To compute the differential rate in the limit of vanishing lepton masses only the functions T_1 , T_2 and T_3 are needed. In the following subsections we give the complete expressions for these scalar functions up to order $1/m^4$. We use the notations

$$\Delta_0 = m_b^2 - 2m_b q \cdot v - m_c^2 + q^2 \tag{B.1}$$

Furthermore we use dimensionless variables according to

$$\frac{m_c^2}{m_b^2} = \rho, \quad \frac{q^2}{m_b^2} \rightarrow q^2, \quad m_b - v \cdot q \rightarrow m_b v \cdot Q, \quad \frac{\Delta_0}{m_b^2} \rightarrow \Delta_1 \tag{B.2}$$

MATHEMATICA notebooks with the corresponding expressions can be obtained from the authors.

B.1 Scalar function T_1

$$\begin{aligned}
 T_1 & = \frac{v \cdot Q M_B}{m_b \Delta_1} \\
 & - \frac{M_B}{m_b^3} \hat{\mu}_\pi^2 \left(-\frac{1}{6\Delta_1} + \frac{v \cdot q (5v \cdot q - 3) - 2q^2}{3\Delta_1^2} - \frac{4v \cdot Q ((v \cdot q)^2 - q^2)}{3\Delta_1^3} \right) \\
 & + \frac{M_B}{3m_b^3} \hat{\mu}_G^2 \left(\frac{3}{\Delta_1} + \frac{-11q^2 + 10(v \cdot q)^2 + 7\rho + 1}{2\Delta_1^2} \right) \\
 & + \frac{M_B}{m_b^4} \hat{\rho}_D^3 \left(\frac{2}{3\Delta_1} + \frac{4(v \cdot q - q^2) + 3\rho + 3}{3\Delta_1^2} - \frac{4((v \cdot q)^2 - q^2) v \cdot Q}{3\Delta_1^3} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{8((v \cdot q)^2 - q^2)(v \cdot Q)^2}{3\Delta_1^4} \\
 & + \frac{M_B}{m_b^4} \hat{\rho}_{LS}^3 \left(\frac{2}{3\Delta_1} + \frac{-4q^2 + 4(v \cdot q)^2 + 3\rho - 1}{3\Delta_1^2} - \frac{4v \cdot Q((v \cdot q)^2 - q^2)}{3\Delta_1^3} \right) \\
 & + \frac{M_B}{m_b^5} s_1 \left(\frac{2v \cdot Q}{3\Delta_1^2} - \frac{2(-30(v \cdot q)^3 + 38(v \cdot q)^2 + 15(q^2 - 4)v \cdot q + 7q^2 + 30)}{15\Delta_1^3} \right. \\
 & \quad - \frac{8(q^2 - (v \cdot q)^2)(-6q^2 + 2v \cdot q(3v \cdot q + 5) + 5(\rho - 2))}{15\Delta_1^4} \\
 & \quad \left. - \frac{16((v \cdot q)^2 - q^2)(-6q^2 + 6(v \cdot q)^2 + 5\rho)v \cdot Q}{15\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^5} s_2 \left(\frac{1 - 3v \cdot q}{6\Delta_1^2} + \frac{(35v \cdot q - 23)q^2 - 2v \cdot q(v \cdot q(25v \cdot q - 39) + 40) + 40}{30\Delta_1^3} \right. \\
 & \quad \left. - \frac{2(q^2 - 6(v \cdot q)^2 + 5\rho + 5)(q^2 - (v \cdot q)^2)}{15\Delta_1^4} + \frac{8v \cdot Q(q^2 - (v \cdot q)^2)^2}{5\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^5} s_3 \left(\frac{v \cdot q + 5}{6\Delta_1^2} + \frac{3(5v \cdot q - 1)q^2 - 2v \cdot q(v \cdot q(15v \cdot q + 11) - 40) - 40}{30\Delta_1^3} \right. \\
 & \quad \left. - \frac{4(3q^2 + (10 - 13v \cdot q)v \cdot q)(q^2 - (v \cdot q)^2)}{15\Delta_1^4} + \frac{8v \cdot Q(q^2 - (v \cdot q)^2)^2}{5\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^5} s_4 \left(-\frac{7v \cdot q + 1}{12\Delta_1^2} + \frac{2(v \cdot q + 1)(q^2 - (v \cdot q)^2)}{3\Delta_1^3} \right. \\
 & \quad \left. - \frac{2(q^2 - (v \cdot q)^2)(q^2 - 2(v \cdot q)^2 + \rho + 1)}{5\Delta_1^4} \right) \\
 & + \frac{M_B}{m_b^5} s_5 \left(\frac{3v \cdot q + 5}{4\Delta_1^2} + \frac{2(q^2(3v \cdot q + 1) - 2v \cdot q(2 - 3v \cdot Qv \cdot q))}{3\Delta_1^3} \right. \\
 & \quad \left. + \frac{4(q^2 - (v \cdot q)^2)(-19q^2 + 13(v \cdot q)^2 + 16\rho + 6)}{15\Delta_1^4} \right) \tag{B.3}
 \end{aligned}$$

B.2 The scalar function T_2

$$\begin{aligned}
 T_2 = & \frac{2M_B}{m_b\Delta_1} - \frac{M_B}{m_b^3} \hat{\mu}_\pi^2 \left(-\frac{5}{3\Delta_1} - \frac{14v \cdot q}{3\Delta_1^2} + \frac{8(q^2 - (v \cdot q)^2)}{3\Delta_1^3} \right) \\
 & + \frac{M_B}{3m_b^3} \hat{\mu}_G^2 \left(-\frac{5}{\Delta_1} + \frac{4 - 10v \cdot q}{\Delta_1^2} \right) \\
 & + \frac{M_B}{m_b^4} \hat{\rho}_D^3 \left(-\frac{4}{3\Delta_1} + \frac{4 - 4v \cdot q}{\Delta_1^2} + \frac{16v \cdot Qv \cdot q}{3\Delta_1^3} - \frac{16v \cdot Q(q^2 - (v \cdot q)^2)}{3\Delta_1^4} \right) \\
 & + \frac{M_B}{m_b^4} \hat{\rho}_{LS}^3 \left(-\frac{4}{3\Delta_1} + \frac{2 - 3v \cdot q}{12\Delta_1^2} - \frac{4(q^2 - 2v \cdot Qv \cdot q)}{3\Delta_1^3} \right) \\
 & + \frac{M_B}{m_b^5} s_1 \left(-\frac{4(3v \cdot q + 5)}{3\Delta_1^2} - \frac{4(-39q^2 + 42(v \cdot q)^2 + 40v \cdot q + 35\rho - 5)}{15\Delta_1^3} \right. \\
 & \quad \left. - \frac{32((6v \cdot q + 5)((v \cdot q)^2 - q^2) + 5v \cdot qp)}{15\Delta_1^4} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{32(q^2 - (v \cdot q)^2)(-6q^2 + 6(v \cdot q)^2 + 5\rho)}{15\Delta_1^5} \\
 & + \frac{M_B}{m_b^5} s_2 \left(\frac{5v \cdot q + 3}{3\Delta_1^2} + \frac{-39q^2 + 62(v \cdot q)^2 + 5\rho + 15}{15\Delta_1^3} \right. \\
 & \quad \left. + \frac{8(6v \cdot q + 5)((v \cdot q)^2 - q^2)}{15\Delta_1^4} + \frac{16(q^2 - (v \cdot q)^2)^2}{5\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^5} s_3 \left(\frac{3v \cdot q - 1}{3\Delta_1^2} + \frac{q^2 + 82(v \cdot q)^2 - 25\rho + 5}{15\Delta_1^3} \right. \\
 & \quad \left. + \frac{128v \cdot q((v \cdot q)^2 - q^2)}{15\Delta_1^4} + \frac{16(q^2 - (v \cdot q)^2)^2}{5\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^5} s_4 \left(\frac{4v \cdot q + 1}{6\Delta_1^2} - \frac{4(q^2 + (5 - 8v \cdot q)v \cdot q)}{15\Delta_1^3} - \frac{4v \cdot Q(q^2 + 2(v \cdot q)^2)}{5\Delta_1^4} \right) \\
 & + \frac{M_B}{m_b^5} s_5 \left(\frac{24v \cdot q + 13}{6\Delta_1^2} + \frac{-94q^2 + 172(v \cdot q)^2 - 20\rho + 20}{15\Delta_1^3} \right. \\
 & \quad \left. + \frac{4((17 - 47v \cdot q)q^2 + 2(v \cdot q)^2(13v \cdot q + 2))}{15\Delta_1^4} \right)
 \end{aligned} \tag{B.4}$$

B.3 The scalar function T_3

$$\begin{aligned}
 T_3 = & \frac{M_B}{m_b^2 \Delta_1} - \frac{M_B}{m_b^4} \hat{\mu}_\pi^2 \left(-\frac{5v \cdot q}{3\Delta_1^2} + \frac{4(q^2 - (v \cdot q)^2)}{3\Delta_1^3} \right) + \frac{M_B}{3m_b^4} \hat{\mu}_G^2 \left(\frac{6 - 5v \cdot q}{\Delta_1^2} \right) \\
 & + \frac{M_B}{m_b^5} \hat{\rho}_D^3 \left(\frac{6 - 4v \cdot q}{3\Delta_1^2} + \frac{4v \cdot Qv \cdot q}{3\Delta_1^3} - \frac{8v \cdot Q(q^2 - (v \cdot q)^2)}{3\Delta_1^4} \right) \\
 & + \frac{M_B}{m_b^5} \hat{\rho}_{LS}^3 \left(\frac{6 - 4v \cdot q}{3\Delta_1^2} - \frac{4(v \cdot Q)^2}{3\Delta_1^3} \right) \\
 & + \frac{M_B}{m_b^6} s_1 \left(\frac{2(q^2 - 6(v \cdot q)^2 + 8v \cdot q - 6)}{3\Delta_1^3} + \frac{16(3v \cdot q + 5)(q^2 - (v \cdot q)^2) - 40v \cdot q\rho}{15\Delta_1^4} \right. \\
 & \quad \left. + \frac{16(q^2 - (v \cdot q)^2)(-6q^2 + 6(v \cdot q)^2 + 5\rho)}{15\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^6} s_2 \left(\frac{5}{3\Delta_1^2} + \frac{-9q^2 + 10(v \cdot q)^2 + 4\rho + 2}{6\Delta_1^3} \right. \\
 & \quad \left. + \frac{4(3v \cdot q + 5)((v \cdot q)^2 - q^2)}{15\Delta_1^4} + \frac{8(q^2 - (v \cdot q)^2)^2}{5\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^6} s_3 \left(-\frac{1}{\Delta_1^2} + \frac{5q^2 + 6((v \cdot q)^2 - \rho)}{6\Delta_1^3} + \right. \\
 & \quad \left. + \frac{52v \cdot q((v \cdot q)^2 - q^2)}{15\Delta_1^4} + \frac{8(q^2 - (v \cdot q)^2)^2}{5\Delta_1^5} \right) \\
 & + \frac{M_B}{m_b^6} s_4 \left(\frac{1}{4\Delta_1^2} - \frac{2(\rho - (v \cdot q)^2)}{3\Delta_1^3} + \frac{4v \cdot q((v \cdot q)^2 - q^2)}{5\Delta_1^4} \right) \\
 & + \frac{M_B}{m_b^6} s_5 \left(\frac{35}{12\Delta_1^2} + \frac{-11q^2 + 12(v \cdot q)^2 + 9\rho - 9}{3\Delta_1^3} + \frac{4(13v \cdot q - 25)((v \cdot q)^2 - q^2)}{15\Delta_1^4} \right)
 \end{aligned} \tag{B.5}$$

C. Decay rates

Finally we list in this appendix the full expressions for the differential and the total decay rate at tree-level up to order $1/m_b^4$. **MATHEMATICA** notebooks with the corresponding expressions can be obtained from the authors.

C.1 Double differential rate $d^2\Gamma/(d\hat{E}_0 dM_X^2)$

We use for this double differential rate the variables

$$\begin{aligned}
 E_0 &= m_b - v \cdot q, & (C.1) \\
 M_X^2 &= (m_b v - q)^2 \\
 \frac{d\Gamma^{(2)}}{d\hat{M}_X^2 d\hat{E}_0} &= \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \sqrt{\hat{E}_0^2 - \hat{M}_X^2} \left\{ \left(16 \left(4\hat{E}_0^2 - 3 \left(\hat{M}_X^2 + 1 \right) \hat{E}_0 + 2\hat{M}_X^2 \right) \right) \delta(\hat{M}_X^2 - \rho) \right. \\
 &+ \frac{8}{3m_b^2} \hat{\mu}_\pi^2 \left(\left(40\hat{E}_0^3 - 2\hat{E}_0^2 \left(10\hat{M}_X^2 + 5\rho + 11 \right) - 3\rho \right. \right. \\
 &\quad \left. \left. + \hat{E}_0 \left(6\rho - 10\hat{M}_X^2 \right) + 5\hat{M}_X^4 + 7(\rho + 1)\hat{M}_X^2 \right) \delta'(\hat{M}_X^2 - \rho) \right. \\
 &\quad \left. + \left(16\hat{E}_0^4 - 4 \left(2\hat{M}_X^2 + \rho + 3 \right) \hat{E}_0^3 - 8\rho\hat{E}_0^2 + 2\hat{M}_X^2 \left(\hat{M}_X^2 - 5\rho \right) \right. \right. \\
 &\quad \left. \left. + \left(-\hat{M}_X^4 + (13\rho + 3)\hat{M}_X^2 + 9\rho \right) \hat{E}_0 \right) \delta''(\hat{M}_X^2 - \rho) \right. \\
 &+ \frac{8}{3m_b^2} \hat{\mu}_G^2 \left(- \left(+40\hat{E}_0^3 - 2\hat{E}_0^2 \left(10\hat{M}_X^2 + 5\rho + 11 \right) \right. \right. \\
 &\quad \left. \left. + \hat{E}_0 \left(6\rho - 10\hat{M}_X^2 \right) + 5\hat{M}_X^4 + 7(\rho + 1)\hat{M}_X^2 - 3\rho \right) \delta'(\hat{M}_X^2 - \rho) \right. \\
 &\quad \left. + \left(\left(8\hat{E}_0^2 - 3 \left(\hat{M}_X^2 + 1 \right) \hat{E}_0 - 2\hat{M}_X^2 \right) \left(\hat{M}_X^2 - \rho \right) \right) \delta''(\hat{M}_X^2 - \rho) \right) \\
 &+ \frac{8}{3m_b^3} \hat{\rho}_D^3 \left(-2 \left(16\hat{E}_0^3 - 2\hat{E}_0^2 \left(4\hat{M}_X^2 + 2\rho + 5 \right) + \hat{E}_0 \left(6\rho - 4\hat{M}_X^2 \right) \right. \right. \\
 &\quad \left. \left. + 2\hat{M}_X^4 + (\rho + 4)\hat{M}_X^2 - 3\rho \right) \delta'(\hat{M}_X^2 - \rho) \right. \\
 &\quad \left. - 4 \left(\hat{E}_0^2 - \hat{M}_X^2 \right) \left(4\hat{E}_0^2 - \left(2\hat{M}_X^2 + \rho + 1 \right) \hat{E}_0 - \hat{M}_X^2 + \rho \right) \delta''(\hat{M}_X^2 - \rho) \right. \\
 &\quad \left. - \frac{1}{3} \left(32\hat{E}_0^5 - 8 \left(2\hat{M}_X^2 + \rho + 3 \right) \hat{E}_0^4 + 8 \left(\rho - 3\hat{M}_X^2 \right) \hat{E}_0^3 \right. \right. \\
 &\quad \left. \left. + 2 \left(5\hat{M}_X^4 + 2(5\rho + 6)\hat{M}_X^2 - 3\rho^2 \right) \hat{E}_0^2 + 2 \left(5\hat{M}_X^4 - 22\rho\hat{M}_X^2 + 9\rho^2 \right) \hat{E}_0 \right. \right. \\
 &\quad \left. \left. - 3 \left(\hat{M}_X^2 + 3 \right) \left(\hat{M}_X^2 - \rho \right)^2 \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \right) \\
 &+ \frac{8}{3m_b^3} \hat{\rho}_{LS}^3 \left(-2 \left(+16\hat{E}_0^3 - 2\hat{E}_0^2 \left(4\hat{M}_X^2 + 2\rho + 5 \right) + \hat{E}_0 \left(6\rho - 4\hat{M}_X^2 \right) \right. \right. \\
 &\quad \left. \left. + 2\hat{M}_X^4 + (\rho + 4)\hat{M}_X^2 - 3\rho \right) \delta'(\hat{M}_X^2 - \rho) \right)
 \end{aligned}$$

$$\begin{aligned}
 & -4 \left(\hat{E}_0^2 - \hat{M}_X^2 \right) \left(4\hat{E}_0^2 - \left(2\hat{M}_X^2 + \rho + 1 \right) \hat{E}_0 - \hat{M}_X^2 + \rho \right) \delta''(\hat{M}_X^2 - \rho) \\
 & + \frac{1}{3} \left(8\hat{E}_0^3 \left(\hat{M}_X^2 - \rho \right) - 2\hat{E}_0^2 \left(\hat{M}_X^4 + 2\hat{M}_X^2 - \rho(\rho + 2) \right) \right. \\
 & \quad \left. - 2\hat{E}_0 \left(\hat{M}_X^4 + 2\rho\hat{M}_X^2 - 3\rho^2 \right) - \hat{M}_X^6 + (6\rho + 1)\hat{M}_X^4 \right. \\
 & \quad \left. + (2 - 5\rho)\rho\hat{M}_X^2 - 3\rho^2 \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \\
 & + \frac{4}{3m_b^4} s_1 \left(-4 \left(24\hat{E}_0^4 - 2 \left(6\hat{M}_X^2 + 3\rho + 7 \right) \hat{E}_0^3 + \left(8\rho - 12\hat{M}_X^2 \right) \hat{E}_0^2 \right. \right. \\
 & \quad \left. \left. + \left(6\hat{M}_X^4 + (3\rho + 8)\hat{M}_X^2 - 3\rho \right) \hat{E}_0 - 2\rho\hat{M}_X^2 \right) \delta''(\hat{M}_X^2 - \rho) \right. \\
 & \quad - \frac{16}{15} \left(\hat{E}_0^2 - \hat{M}_X^2 \right) \left(24\hat{E}_0^3 - 6 \left(2\hat{M}_X^2 + \rho + 1 \right) \hat{E}_0^2 \right. \\
 & \quad \left. + \left(5\rho - 9\hat{M}_X^2 \right) \hat{E}_0 + \hat{M}_X^2 \left(2\hat{M}_X^2 + \rho + 1 \right) \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \\
 & \quad - \frac{1}{15} \left(192\hat{E}_0^6 - 48 \left(2\hat{M}_X^2 + \rho + 3 \right) \hat{E}_0^5 + \left(48\rho - 176\hat{M}_X^2 \right) \hat{E}_0^4 \right. \\
 & \quad \left. + 8 \left(9\hat{M}_X^4 + (17\rho + 21)\hat{M}_X^2 - 5\rho^2 \right) \hat{E}_0^3 + 4 \left(19\hat{M}_X^4 - 64\rho\hat{M}_X^2 + 25\rho^2 \right) \hat{E}_0^2 \right. \\
 & \quad \left. - \left(21\hat{M}_X^6 + (69 - 2\rho)\hat{M}_X^4 + 5(\rho - 18)\rho\hat{M}_X^2 + 45\rho^2 \right) \hat{E}_0 \right. \\
 & \quad \left. - 2\hat{M}_X^2 \left(\hat{M}_X^4 - 14\rho\hat{M}_X^2 + 5\rho^2 \right) \right) \delta^{(4)}(\hat{M}_X^2 - \rho) \\
 & + \frac{1}{3m_b^4} s_2 \left(4 \left(40\hat{E}_0^4 - 2 \left(10\hat{M}_X^2 + 5\rho + 13 \right) \hat{E}_0^3 - 4 \left(\hat{M}_X^2 + 2\rho \right) \hat{E}_0^2 \right. \right. \\
 & \quad \left. \left. + \left(2\hat{M}_X^4 + (19\rho + 8)\hat{M}_X^2 + 9\rho \right) \hat{E}_0 - 10\rho\hat{M}_X^2 \right) \delta''(\hat{M}_X^2 - \rho) \right. \\
 & \quad + \frac{8}{15} \left(48\hat{E}_0^5 - 12\hat{E}_0^4 \left(2\hat{M}_X^2 + \rho + 1 \right) - 6\hat{E}_0^3 \left(11\hat{M}_X^2 + 5\rho \right) \right. \\
 & \quad \left. + 2\hat{E}_0^2 \left(14\hat{M}_X^4 + (17\rho + 7)\hat{M}_X^2 + 5\rho(\rho + 1) \right) + 6\hat{E}_0 \left(3\hat{M}_X^4 + 5\rho^2 \right) \right. \\
 & \quad \left. - 4\hat{M}_X^6 - (7\rho + 2)\hat{M}_X^4 + 5(1 - 5\rho)\rho\hat{M}_X^2 - 15\rho^2 \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \\
 & \quad + \frac{1}{15} \left(192\hat{E}_0^6 - 48 \left(2\hat{M}_X^2 + \rho + 3 \right) \hat{E}_0^5 - 16 \left(11\hat{M}_X^2 + 7\rho \right) \hat{E}_0^4 \right. \\
 & \quad \left. + 24 \left(3\hat{M}_X^4 + (9\rho + 7)\hat{M}_X^2 + 5\rho \right) \hat{E}_0^3 + 4 \left(19\hat{M}_X^4 - 34\rho\hat{M}_X^2 + 15\rho^2 \right) \hat{E}_0^2 \right. \\
 & \quad \left. - 3 \left(7\hat{M}_X^6 + (26\rho + 23)\hat{M}_X^4 + 5\rho(3\rho + 2)\hat{M}_X^2 + 15\rho^2 \right) \hat{E}_0 \right. \\
 & \quad \left. - 2\hat{M}_X^2 \left(\hat{M}_X^4 - 34\rho\hat{M}_X^2 - 15\rho^2 \right) \right) \delta^{(4)}(\hat{M}_X^2 - \rho) \\
 & + \frac{1}{3m_b^4} s_3 \left(4 \left(24\hat{E}_0^4 - 2 \left(6\hat{M}_X^2 + 3\rho + 7 \right) \hat{E}_0^3 + \left(8\rho - 12\hat{M}_X^2 \right) \hat{E}_0^2 \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left(6\hat{M}_X^4 + (3\rho + 8)\hat{M}_X^2 - 3\rho \right) \hat{E}_0 - 2\rho\hat{M}_X^2 \Big) \delta''(\hat{M}_X^2 - \rho) \\
& + \frac{8}{15} \left(208\hat{E}_0^5 - 4\hat{E}_0^4 \left(26\hat{M}_X^2 + 13\rho + 33 \right) - 2\hat{E}_0^3 \left(23\hat{M}_X^2 + 65\rho \right) \right. \\
& + 2\hat{E}_0^2 \left(14\hat{M}_X^4 + (67\rho + 27)\hat{M}_X^2 + 15\rho(\rho + 3) \right) \\
& \left. - 2\hat{E}_0 \left(21\hat{M}_X^4 - 50\rho\hat{M}_X^2 + 45\rho^2 \right) \right. \\
& \left. + 16\hat{M}_X^6 + (18 - 67\rho)\hat{M}_X^4 + 15(\rho - 5)\rho\hat{M}_X^2 + 45\rho^2 \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \\
& + \frac{1}{15} \left(192\hat{E}_0^6 - 48 \left(2\hat{M}_X^2 + \rho + 3 \right) \hat{E}_0^5 - 16 \left(\hat{M}_X^2 + 17\rho \right) \hat{E}_0^4 \right. \\
& + 16 \left(2\hat{M}_X^4 + (11\rho + 3)\hat{M}_X^2 + 5\rho(\rho + 3) \right) \hat{E}_0^3 - 4 \left(21\hat{M}_X^4 - 26\rho\hat{M}_X^2 + 5\rho^2 \right) \hat{E}_0^2 \\
& + \left(19\hat{M}_X^6 + (51 - 38\rho)\hat{M}_X^4 - 25\rho(5\rho + 6)\hat{M}_X^2 - 45\rho^2 \right) \hat{E}_0 \\
& \left. - 2\hat{M}_X^2 \left(\hat{M}_X^4 + 6\rho\hat{M}_X^2 - 55\rho^2 \right) \right) \delta^{(4)}(\hat{M}_X^2 - \rho) \\
& + \frac{2}{3m_b^4} s_4 \left(\left(32\hat{E}_0^4 - 8 \left(2\hat{M}_X^2 + \rho + 3 \right) \hat{E}_0^3 + 8 \left(\hat{M}_X^2 - 3\rho \right) \hat{E}_0^2 \right. \right. \\
& \left. \left. + \left(-5\hat{M}_X^4 + (29\rho + 3)\hat{M}_X^2 + 21\rho \right) \hat{E}_0 + 2\hat{M}_X^2 \left(\hat{M}_X^2 - 9\rho \right) \right) \delta''(\hat{M}_X^2 - \rho) \right. \\
& + \frac{8}{15} \left(24\hat{E}_0^5 - 6\hat{E}_0^4 \left(2\hat{M}_X^2 + \rho + 1 \right) + 2\hat{E}_0^3 \left(\hat{M}_X^2 - 25\rho \right) \right. \\
& + \hat{E}_0^2 \left(-6\hat{M}_X^4 + (47\rho - 13)\hat{M}_X^2 - 5(\rho - 5)\rho \right) + 2\hat{E}_0 \left(2\hat{M}_X^4 - 5\rho\hat{M}_X^2 + 15\rho^2 \right) \\
& \left. + 3\hat{M}_X^6 + (4 - 11\rho)\hat{M}_X^4 + 5(1 - 2\rho)\rho\hat{M}_X^2 - 15\rho^2 \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \\
& - \frac{4}{15} \left(\hat{E}_0^2 - \hat{M}_X^2 \right) \left(\hat{M}_X^2 - \rho \right) \left(-6\hat{E}_0^2 + 5 \left(\hat{M}_X^2 - \rho \right) \hat{E}_0 + \hat{M}_X^2 + 5\rho \right) \delta^{(4)}(\hat{M}_X^2 - \rho) \\
& + \frac{2}{3m_b^4} s_5 \left(\left(192\hat{E}_0^4 - 16 \left(6\hat{M}_X^2 + 3\rho + 7 \right) \hat{E}_0^3 - 8 \left(13\hat{M}_X^2 - 9\rho \right) \hat{E}_0^2 \right. \right. \\
& \left. \left. + \left(51\hat{M}_X^4 + (21\rho + 67)\hat{M}_X^2 - 27\rho \right) \hat{E}_0 + 2\hat{M}_X^2 \left(\hat{M}_X^2 - 9\rho \right) \right) \delta''(\hat{M}_X^2 - \rho) \right. \\
& + \frac{8}{15} \left(104\hat{E}_0^5 - 2\hat{E}_0^4 \left(26\hat{M}_X^2 + 13\rho + 33 \right) - 8\hat{E}_0^3 \left(16\hat{M}_X^2 - 5\rho \right) \right. \\
& + \hat{E}_0^2 \left(74\hat{M}_X^4 + (87 - 23\rho)\hat{M}_X^2 + 15\rho(3\rho - 1) \right) + \hat{E}_0 \left(-6\hat{M}_X^4 + 80\rho\hat{M}_X^2 - 90\rho^2 \right) \\
& \left. - 7\hat{M}_X^6 - (11\rho + 6)\hat{M}_X^4 - 45\rho\hat{M}_X^2 + 45\rho^2 \right) \delta^{(3)}(\hat{M}_X^2 - \rho) \\
& - \frac{2}{15} \left(108 \left(\hat{M}_X^2 - \rho \right) \hat{E}_0^4 - 30 \left(2\hat{M}_X^4 - 3\rho\hat{M}_X^2 + \hat{M}_X^2 + (\rho - 1)\rho \right) \hat{E}_0^3 \right. \\
& \left. + \left(-86\hat{M}_X^4 + 16\rho\hat{M}_X^2 + 70\rho^2 \right) \hat{E}_0^2 + 15 \left(3\hat{M}_X^6 + (1 - 4\rho)\hat{M}_X^4 + \rho^2\hat{M}_X^2 - \rho^2 \right) \hat{E}_0 \right.
\end{aligned}$$

$$+8\hat{M}_X^2 \left(\hat{M}_X^4 + 4\rho\hat{M}_X^2 - 5\rho^2 \right) \delta^{(4)}(\hat{M}_X^2 - \rho) \Big\} \quad (\text{C.2})$$

C.2 Differential rate $d\Gamma/dy$

We use in this differential rate the dimensionless variable

$$y = 2\frac{E_e}{m_b}. \quad (\text{C.3})$$

Furthermore x is a short hand notation for $x = \frac{y}{1-y}(1-y-\rho)$. $\delta^{(n)}(x)$ means therefore the n^{th} derivative of the delta function with respect to its argument x . To rewrite the argument and derivative of the delta function linear in y , we use

$$\delta^{(n)}(g(y)) = \left(\frac{1}{g'(y)} \frac{d}{dy} \right)^n \sum_i \frac{1}{|g'(y_i)|} \delta(y - y_i)$$

where

$$\left(\frac{1}{g'(y)} \frac{d}{dy} \right)^n = \underbrace{\frac{1}{g'(y)} \frac{d}{dy} \cdots \frac{1}{g'(y)} \frac{d}{dy}}_{n \text{ times}}$$

$$\begin{aligned} \frac{d\Gamma}{dy} = & \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left\{ 2y^2 \left(\frac{(y-3)\rho^3}{(y-1)^3} - \frac{3\rho^2}{(y-1)^2} - 3\rho - 2y + 3 \right) \right. \\ & - \frac{2y^3}{3m_b^2} \hat{\mu}_\pi^2 \left(\frac{2(y^2 - 5y + 10)\rho^3}{(y-1)^5} + \frac{(15-6y)\rho^2}{(y-1)^4} + 5 \right) \\ & + \frac{2y^2}{3m_b^2} \hat{\mu}_G^2 \left(\frac{5(y^2 - 4y + 6)\rho^3}{(y-1)^4} - \frac{9(y-2)\rho^2}{(y-1)^3} + \frac{6(2y-3)\rho}{(y-1)^2} + 5y + 6 \right) \\ & + \frac{2}{3m_b^3} \hat{\rho}_D^3 \left(\frac{y^2(5y^4 - 30y^3 + 75y^2 - 84y + 54)\rho^3}{(y-1)^6} \right. \\ & \quad + \frac{y^2(-5y^3 + 25y^2 - 50y + 42)\rho^2}{(y-1)^5} + \frac{2y^2(2y^2 - 3y - 3)\rho}{(y-1)^3} \\ & \quad + \frac{2y^2(2y^2 - 5y + 9)}{y-1} - \frac{(\rho+1)(\rho+2)(\rho-1)^4}{\rho^2} \delta(y+\rho-1) \\ & \quad \left. + \left(\frac{\rho^4(-y^2 + 3y + \rho - 2)}{(y-1)^8} - \frac{\rho^3 y}{(y-1)^5} + \frac{(y-2)\rho^2}{(y-1)^3} + \frac{\rho}{y-1} \right) y^4 \delta'(x) \right) \\ & + \frac{2y^2}{3m_b^3} \hat{\rho}_{LS}^3 \left(\frac{(y^3 - 5y^2 + 10y + 6)\rho^3}{(y-1)^5} + \frac{3(y^2 - 4y + 6)\rho^2}{(y-1)^4} + \frac{6(2y-3)\rho}{(y-1)^2} + 4y + 6 \right) \\ & + \frac{1}{15m_b^4} s_1 \left(- \frac{6y^2(17y^5 - 119y^4 + 357y^3 - 595y^2 + 520y - 240)\rho^3}{(y-1)^7} \right. \\ & \quad + \frac{2y^2(43y^4 - 258y^3 + 645y^2 - 760y + 420)\rho^2}{(y-1)^6} \\ & \quad + \frac{20y^2(4y^3 - 15y^2 + 20y - 12)\rho}{(y-1)^4} + \frac{40y^2(2y^2 - 4y + 3)}{(y-1)^2} \\ & \quad \left. + \frac{2(3\rho^4 + 10\rho^3 - 14\rho^2 - 34\rho - 45)(\rho-1)^3}{\rho^3} \delta(y+\rho-1) \right) \end{aligned}$$

$$\begin{aligned}
 & + \left(-\frac{3(y-19)\rho^5}{(y-1)^9} + \frac{(-2y^2-28y+75)\rho^4}{(y-1)^8} - \frac{2(5y+8)\rho^3y}{(y-1)^6} \right. \\
 & \quad \left. - \frac{2(11y+2)\rho^2y}{(y-1)^4} - \frac{3(y^2+4y+5)\rho}{(y-1)^2} - 3 \right) y^4 \delta'(x) \\
 & + \left(+\frac{3\rho^6}{(y-1)^{10}} + \frac{(5-2y)\rho^5}{(y-1)^9} - \frac{2\rho^4y}{(y-1)^7} \right. \\
 & \quad \left. - \frac{2\rho^3y}{(y-1)^5} + \frac{(3y-5)\rho^2}{(y-1)^3} + \frac{3\rho}{y-1} \right) y^5 \delta''(x) \\
 & + \frac{1}{60m_b^4} s_2 \left(\frac{6y^2(27y^5-189y^4+567y^3-945y^2+820y-340)\rho^3}{(y-1)^7} \right. \\
 & \quad - \frac{6y^2(21y^4-126y^3+315y^2-400y+220)\rho^2}{(y-1)^6} + \frac{40y^3(5y^2-16y+14)\rho}{(y-1)^4} \\
 & \quad + \frac{40(y-2)y^3}{(y-1)^2} - \frac{2(3\rho^3+23\rho^2-11\rho+45)(\rho-1)^4}{\rho^3} \delta(y+\rho-1) \\
 & \quad + \left(\frac{3(y-19)\rho^5}{(y-1)^9} + \frac{(12y^2+128y-185)\rho^4}{(y-1)^8} - \frac{2(15y^2+32y-105)\rho^3}{(y-1)^6} \right. \\
 & \quad \left. + \frac{2(6y^2-8y-45)\rho^2}{(y-1)^4} + \frac{(3y^2+12y+5)\rho}{(y-1)^2} + 3 \right) y^4 \delta'(x) \\
 & + \left(-\frac{3\rho^6}{(y-1)^{10}} + \frac{3(4y-5)\rho^5}{(y-1)^9} + \frac{6(5-3y)\rho^4}{(y-1)^7} \right. \\
 & \quad \left. + \frac{6(2y-5)\rho^3}{(y-1)^5} - \frac{3(y-5)\rho^2}{(y-1)^3} - \frac{3\rho}{y-1} \right) y^5 \delta''(x) \\
 & + \frac{1}{60m_b^4} s_3 \left(\frac{2y^2(-29y^5+203y^4-609y^3+1015y^2-1240y+480)\rho^3}{(y-1)^7} \right. \\
 & \quad + \frac{2y^2(7y^4-42y^3+105y^2-400y+240)\rho^2}{(y-1)^6} \\
 & \quad - \frac{40y^2(3y^3-11y^2+14y-3)\rho}{(y-1)^4} - \frac{40y^2(4y^2-14y+9)}{(y-1)^2} \\
 & \quad + \frac{2(17\rho^2+34\rho+45)(\rho-1)^5}{\rho^3} \delta(y+\rho-1) \\
 & \quad + \left(-\frac{(37y+17)\rho^5}{(y-1)^9} + \frac{(52y^2-32y-65)\rho^4}{(y-1)^8} + \frac{2(5y^2-12y+45)\rho^3}{(y-1)^6} \right. \\
 & \quad \left. - \frac{2(14y^2-32y+25)\rho^2}{(y-1)^4} + \frac{(3y^2-28y+5)\rho}{(y-1)^2} + 3 \right) y^4 \delta'(x) \\
 & + \left(-\frac{3\rho^6}{(y-1)^{10}} + \frac{3(4y-5)\rho^5}{(y-1)^9} + \frac{6(5-3y)\rho^4}{(y-1)^7} \right. \\
 & \quad \left. + \frac{6(2y-5)\rho^3}{(y-1)^5} - \frac{3(y-5)\rho^2}{(y-1)^3} - \frac{3\rho}{y-1} \right) y^5 \delta''(x) \\
 & + \frac{1}{10m_b^4} s_4 \left(\frac{2y^2(y^4-6y^3+15y^2-20y-10)\rho^3}{(y-1)^6} - \frac{y^2(7y^3-35y^2+55y+45)\rho^2}{(y-1)^5} \right)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{2y^2(2y^3-5y^2+15)\rho}{(y-1)^4} + 5y^2 + \frac{2(\rho-2)(\rho+1)(\rho-1)^4}{\rho^2} \delta(y+\rho-1) \\
& + \left(-\frac{2\rho^5}{(y-1)^8} + \frac{6(y-2)\rho^4}{(y-1)^7} - \frac{2(3y^2-13y+12)\rho^3}{(y-1)^6} \right. \\
& \quad \left. + \frac{2(y^2-7y+10)\rho^2}{(y-1)^4} + \frac{2(y-3)\rho}{(y-1)^2} \right) y^4 \delta'(x) \\
& + \frac{1}{30m_b^4} s_5 \left(\frac{4y^2(19y^4-114y^3+285y^2-380y+60)\rho^3}{(y-1)^6} \right. \\
& \quad + \frac{y^2(-11y^3+55y^2-65y+285)\rho^2}{(y-1)^5} + \frac{2y^2(74y^3-215y^2+120y+165)\rho}{(y-1)^4} \\
& \quad + \frac{15(19-11y)y^2}{y-1} + \frac{2(13\rho^2-3\rho+74)(\rho-1)^4}{\rho^2} \delta(y+\rho-1) \\
& \quad \left. + \left(-\frac{26\rho^5}{(y-1)^8} + \frac{2(19y-8)\rho^4}{(y-1)^7} + \frac{2(y^2-31y+54)\rho^3}{(y-1)^6} \right. \right. \\
& \quad \left. \left. - \frac{2(7y^2-39y+80)\rho^2}{(y-1)^4} + \frac{2(31-7y)\rho}{(y-1)^2} \right) y^4 \delta'(x) \right\} \tag{C.4}
\end{aligned}$$

C.3 Total decay rate Γ

$$\begin{aligned}
\Gamma = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left\{ -\rho^4 + 8\rho^3 - 12\log(\rho)\rho^2 - 8\rho + 1 \right. \\
\quad - \frac{1}{2m_b^2} \hat{\mu}_\pi^2 \left(-\rho^4 + 8\rho^3 - 12\log(\rho)\rho^2 - 8\rho + 1 \right) \\
\quad + \frac{1}{2m_b^2} \hat{\mu}_G^2 \left(-5\rho^4 + 24\rho^3 - 12(\log(\rho)+2)\rho^2 + 8\rho - 3 \right) \\
\quad + \frac{2}{3m_b^3} \hat{\rho}_D^3 \left(-5\rho^4 + 16\rho^3 - 12\rho^2 - 16\rho + 12\log(\rho) + 17 \right) \\
\quad + \frac{8}{9m_b^4} s_1 \left(9\rho^4 - 20\rho^3 + 9\rho^2 + 6\log(\rho) + 2 \right) \\
\quad + \frac{1}{9m_b^4} s_2 \left(-27\rho^4 + 76\rho^3 - 72\rho^2 + 36\rho - 12\log(\rho) - 13 \right) \\
\quad + \frac{4}{9m_b^4} s_3 \left(3\rho^4 - 7\rho^3 + 9\rho^2 - 21\rho + 4(3\log(\rho)+4) \right) \\
\quad \left. + \frac{1}{3m_b^4} s_5 \left(-5\rho^4 + 16\rho^3 - 12\rho^2 - 16\rho + 12\log(\rho) + 17 \right) \right\} \tag{C.5}
\end{aligned}$$

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